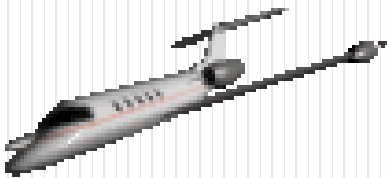


Aircraft Dynamics

First order and Second order system

Prepared
by



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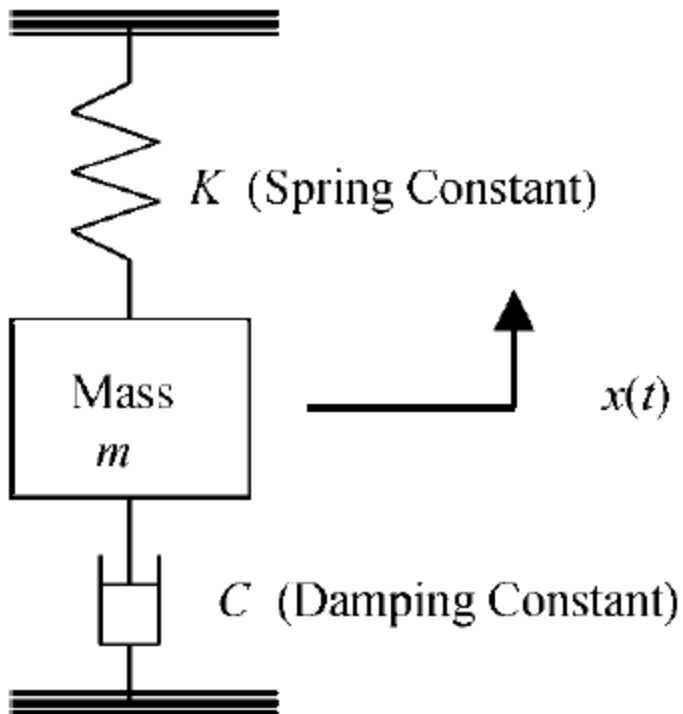
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- Aircraft dynamic stability focuses on the time history of aircraft motion after the aircraft is disturbed from an equilibrium or trim condition.
- This motion may be **first order (exponential response)** or **second order (oscillatory response)**, and will have
- Positive dynamic stability (aircraft returns to the trim condition as time goes to infinity).
- Neutral dynamic stability (aircraft neither returns to trim nor diverges further from the disturbed condition).
- Dynamic instability (aircraft diverges from the trim condition and the disturbed condition as time goes to infinity).

- The study of dynamic stability is important to understanding aircraft handling qualities and the design features that make an airplane fly well or not as well while performing specific mission tasks.
- The differential equations that define the aircraft equations of motion (EOM) form the starting point for the study of dynamic stability.

Mass–Spring–Damper System and Classical Solutions of Ordinary Differential Equations



- The mass–spring–damper system illustrated in Figure. It provides a starting point for analysis of system dynamics and aircraft dynamic stability.
- This is an excellent model to begin the understanding of dynamic response.

- We will first develop an expression for the sum of forces in the vertical direction.
- Notice that $x(t)$ is defined as positive for an upward displacement and that the zero position is chosen as the point where the system is initially at rest or at equilibrium. We know that

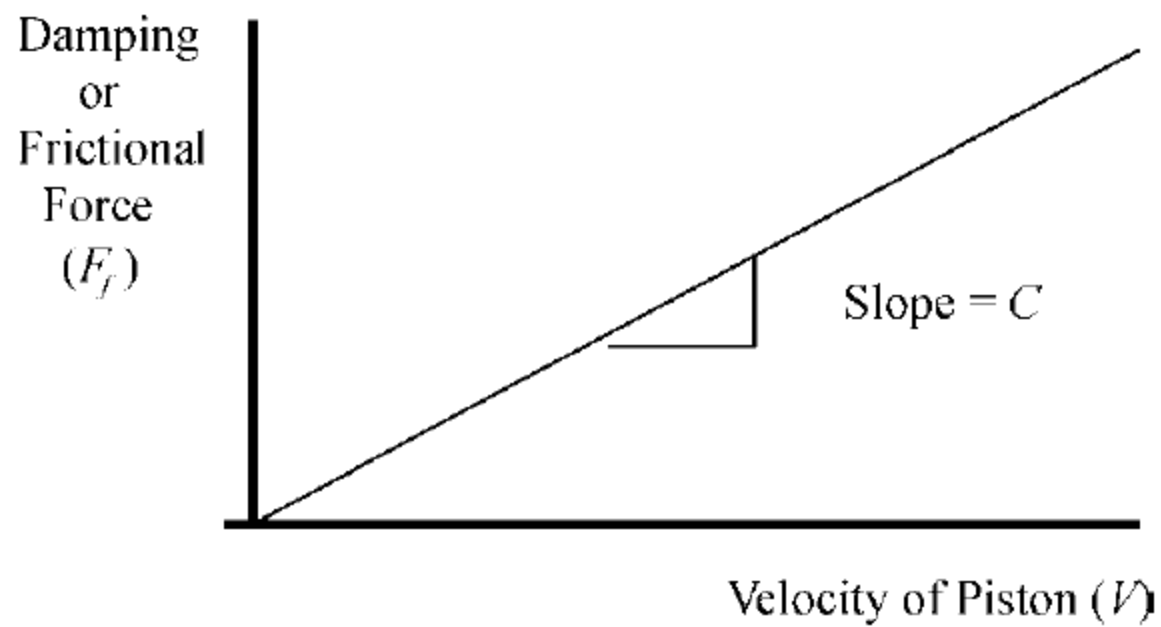
$$\sum F_x = m \frac{d^2 x}{dt^2}$$

- There are two forces acting on the mass, the damping force, and the spring force.

- For the damping or frictional force (F_f) this can be approximated by a linear relationship of damping force as a function of velocity or dx/dt
- A damper can be thought of as a “shock absorber” with a piston moving up and down inside a cylinder. The piston is immersed in a fluid and the fluid is displaced through a small orifice to provide a resistance force directly proportional to the velocity of the piston.
- This resistance force (F_f) can be expressed as

$$F_f = CV$$

where C is the slope



- The spring force (F_s) is directly proportional to the displacement (x) of the mass and can be represented as

$$F_s = Kx$$

- where **K is the spring constant**. If the mass is displaced in the positive x direction, both the damping and spring forces act in a direction opposite to this displacement and can be represented by

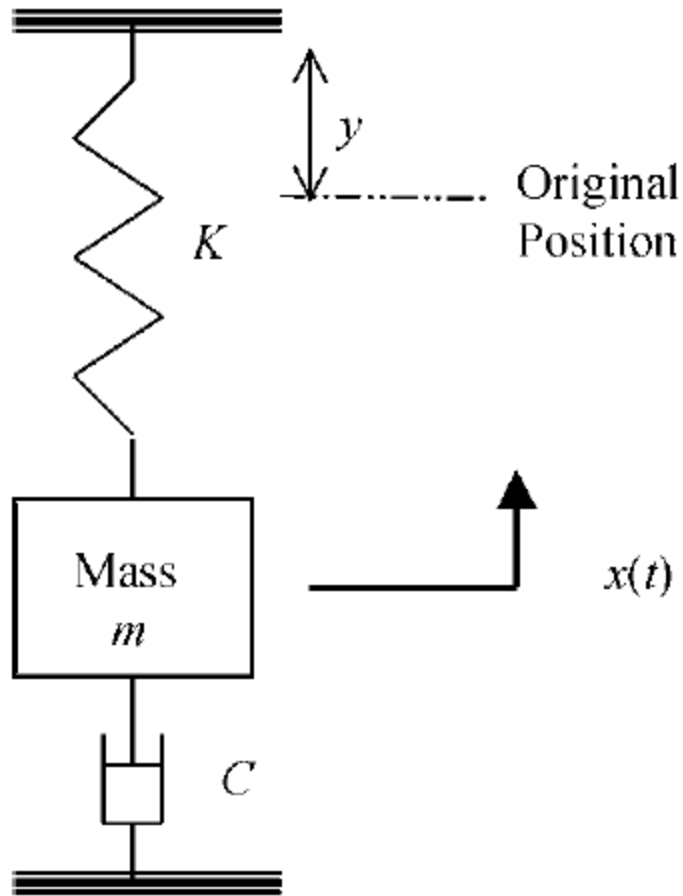
$$F_f + F_s = -CV - Kx$$

$$V = \frac{dx}{dt}$$

$$m\left(\frac{d^2x}{dt^2}\right) = -C\left(\frac{dx}{dt}\right) - Kx$$

$$m\left(\frac{d^2x}{dt^2}\right) + C\left(\frac{dx}{dt}\right) + Kx = 0$$

- which is the differential equation for the mass–spring damper system with zero initial displacement ($X = 0$).



- If we initially stretch the spring from its original position by a distance y as shown in Figure, we build in a forcing function that must be added to the above equation.

$$m\left(\frac{d^2x}{dt^2}\right) + C\left(\frac{dx}{dt}\right) + Kx = Ky$$

- This is the differential equation for the spring–mass–damper system with applied force externally. At this point, we should observe that if the mass is free to move, it will obtain a steady-state condition (a new equilibrium location) when d^2x/dt^2 and dx/dt equal zero and the new equilibrium position will be $x = y$.
- But the system will take certain amount of time to come back its original position because of the presence of spring and damper.
- It is also representative of aircraft motion and that is why we are investigating it in depth.

- A special case spring–mass–damper system where the mass is very small or negligible compared to the size of the spring and damper. We will call such a system a **massless or first-order** (referring to the order of the highest derivative) system.
- The following differential equation results when the mass is set equal to zero.

$$C\left(\frac{dx}{dt}\right) + Kx = Ky$$

$$Px = \frac{dx}{dt} \quad P^2x = \frac{d^2x}{dt^2} \quad \frac{x}{p} = \int x dt$$

$$C\left(\frac{dx}{dt}\right) + Kx = 0$$

$$CPx + Kx = 0$$

$$(CP + K)x = 0$$

$$P = -\frac{K}{C}$$

- The homogeneous solution is then of the form

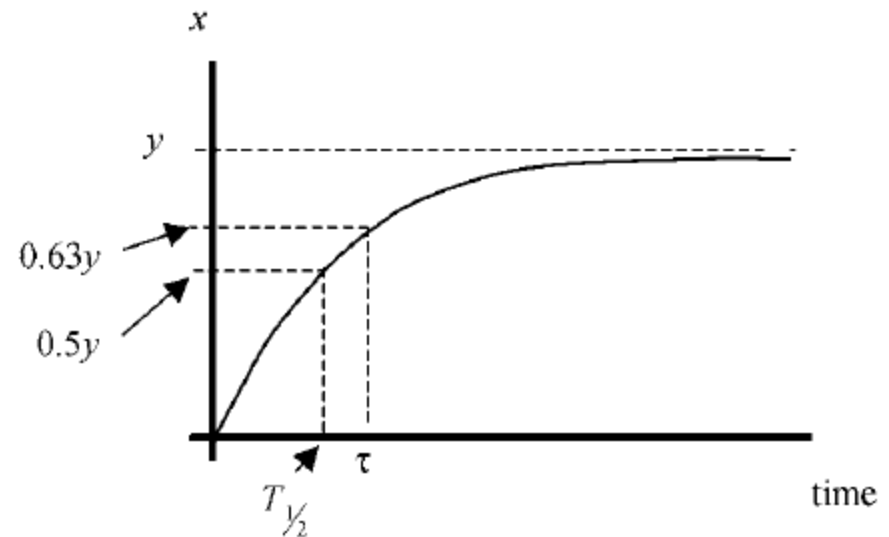
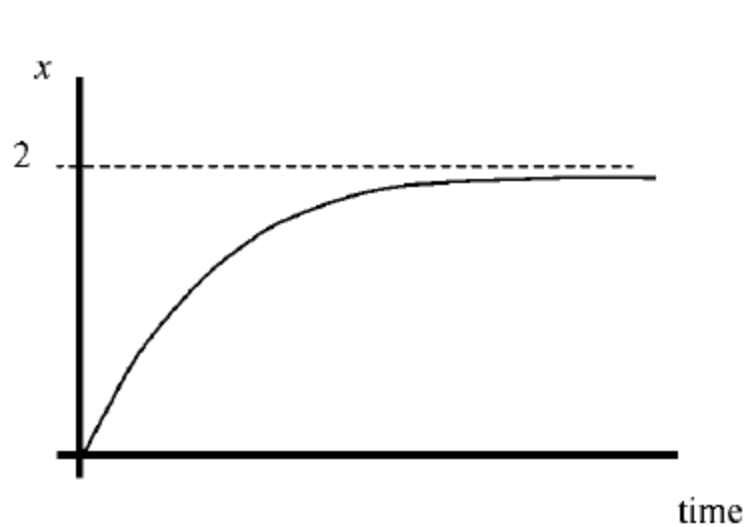
$$\begin{aligned}x(t) &= C_1 e^{pt} \\ &= C_1 e^{(-K/C)t}\end{aligned}$$

- where C_1 is determined from initial conditions. The homogeneous solution will also be called **the transient solution** when we are dealing with aircraft response.

Time constant

- The lag time associated with this rise to the steady-state value is an important consideration in determining the acceptability of the response from an aircraft handling qualities standpoint.

$$x(t) = y(1 - e^{(-K/C)t})$$



- This lag time is typically quantified with the time constant τ which is a measure of the time it takes to achieve 63.2% of the steady-state value.

- Why did we pick 63.2%? If we let, $t = \frac{C}{K}$ our first-order response to a step input becomes

$$\begin{aligned}x(t) &= y(1 - e^{(-K/C)(C/K)}) \\&= y(1 - e^{(-1)}) \\&= y(1 - 0.368) \\&= 0.632y\end{aligned}$$

- The time constant becomes an easy value to determine because

$$\tau = \frac{1}{P}$$

Time to half and double amplitude.

- Another measure of the lag time associated with a systems response is the time to half amplitude ($T_{1/2}$).
- Referring the above figure it is simply the time it takes to achieve 50% of the steady state value.

$$T_{1/2} = \tau(\ln 2) = 0.693\tau$$

- For unstable first order systems ($P > 0$), a measure used as an indication of the instability is the time to double amplitude (T_2).
- T_2 is the time it takes for the response to achieve twice the amplitude of an input disturbance

$$T_2 = \frac{\ln 2}{P} = \frac{0.693}{P}$$

- Note also, because τ is equal to C/K for the spring–mass–damper system.
- τ will increase (meaning a slower responding system) for an increase in damping constant (C)
- τ will decrease (meaning a faster responding system) for an increase in the spring constant (K).
- This should make sense when thinking about the physical dynamics of the system.

Second order system

- Spring–mass–damper system where the mass provides significant inertial effects.
- The second order Spring-mass-damper has been written as

$$M\ddot{x} + C\dot{x} + Kx = Ky \quad (or) \quad (MP^2 + CP + K)x = 0$$

- We can solve the roots by using quadratic equation formula

$$P_{1,2} = -\frac{C}{2M} \pm i \frac{\sqrt{4KM - C^2}}{2M} = a \pm ib$$

Three cases must be considered based on the sign of the expression

- Case 1: $4KM > C^2$ (or) Two real unequal roots
Over damped system (no oscillations)

$$x(t) = C_1 e^{P_1 t} + C_2 t e^{P_1 t}$$

- Case 2: $C^2 = 4KM$ (or) Two real repeated roots
Critically damped system (no oscillations)

$$x(t) = e^{at} [C_1 \sin bt + C_2 \cos bt]$$

- Case 3: $4KM > C^2$ (or) Two real repeated roots

Under damped system (with oscillations) system.

$$P_{1,2} = -\frac{C}{2M} \pm i \frac{\sqrt{4KM - C^2}}{2M} = a \pm ib$$

General solution $x(t) = e^{at} [C_1 \sin bt + C_2 \cos bt]$

Second order system

- We can rewrite the second order system in the form of

$$\ddot{x} + 2\delta\omega_N\dot{x} + \omega_N^2x = \omega_N^2y$$

where δ – damping ratio

ω_N – natural frequency

- The damping ratio provides an indication of the system damping and will fall between -1 and 1.
- For stable systems, the damping ratio will be between 0 and 1.
- For this case, the higher the damping ratio, the more damping is present in the system.

- Notice that the number of overshoots/undershoots varies inversely with the damping ratio.
- The natural frequency is the frequency (in rad/s) that the system would oscillate at if there were no damping.

Natural Frequency

- The natural frequency is the frequency (in rad/s) that **the system would oscillate at if there were no damping**. It represents the highest frequency that the system is capable of, but it is not the frequency that the system actually oscillates at if damping is present.
- For the mass–spring–damper system the natural frequency oscillation

$$\omega_N = \sqrt{K / M}$$

Damped frequency

- The damped frequency (WD) represents the frequency (in rad/s) that the system **actually oscillates at with damping present**.
- We can use the quadratic formula to solve for the roots of the homogeneous form of the equation.

$$P_{1,2} = -\zeta\omega_N \pm i\omega_N\sqrt{1 - \zeta^2} = -\zeta\omega_N \pm i\omega_D = a \pm ib$$

$$\omega_D = \omega_N\sqrt{1 - \zeta^2} = \text{Damped Frequency}$$

- The time constant of a second order system

$$\tau = \frac{1}{\xi\omega_n}$$

- Notice that larger the value of $\xi\omega_n$ will smaller time constant will increases faster response.
- It is similar to calculate the time constant of a first order system

$$\tau = -\frac{1}{pole}$$

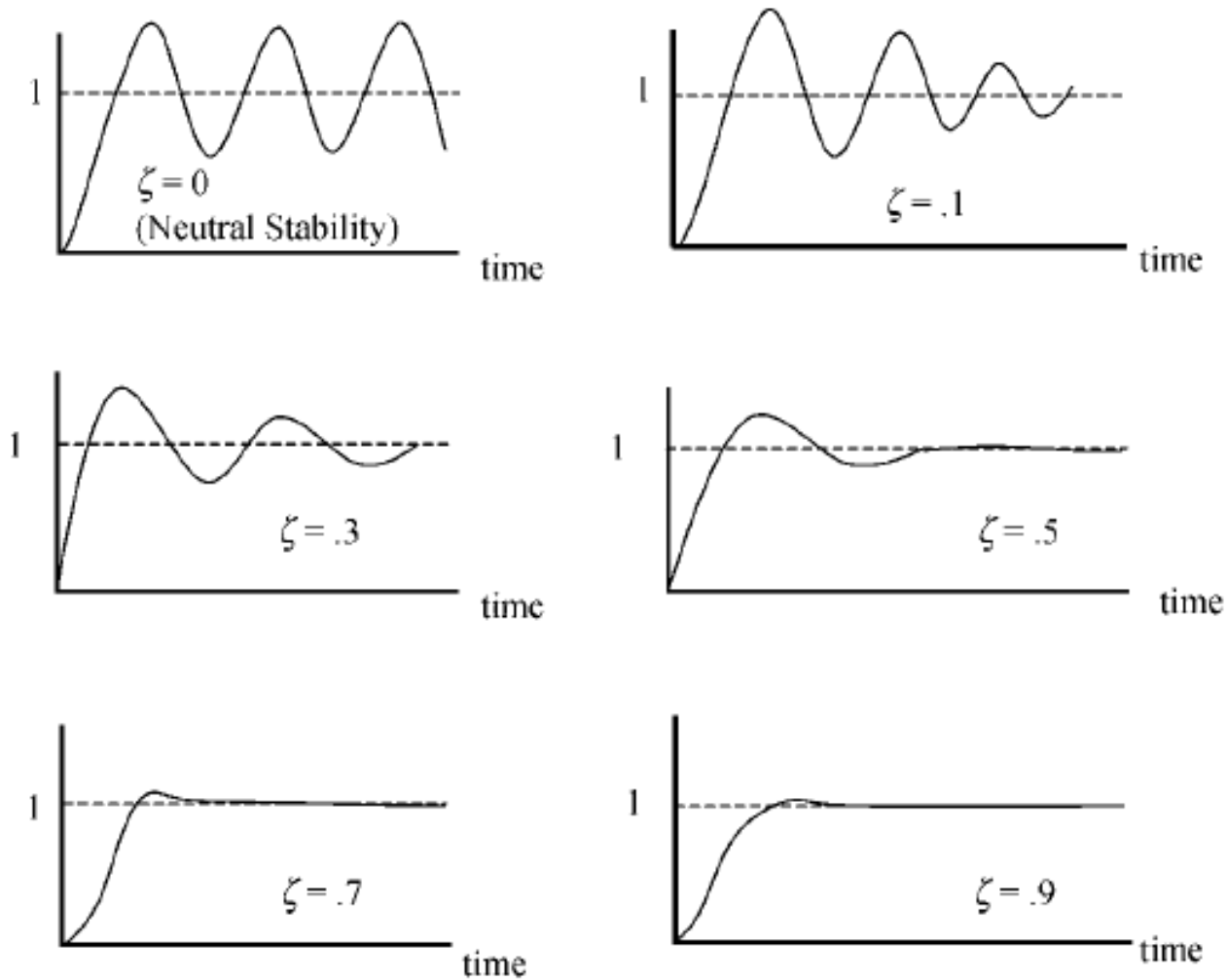


Figure presents a family of second order responses to a unit step ($y=1$) input, which show the influence of damping ratio.

Period of oscillation

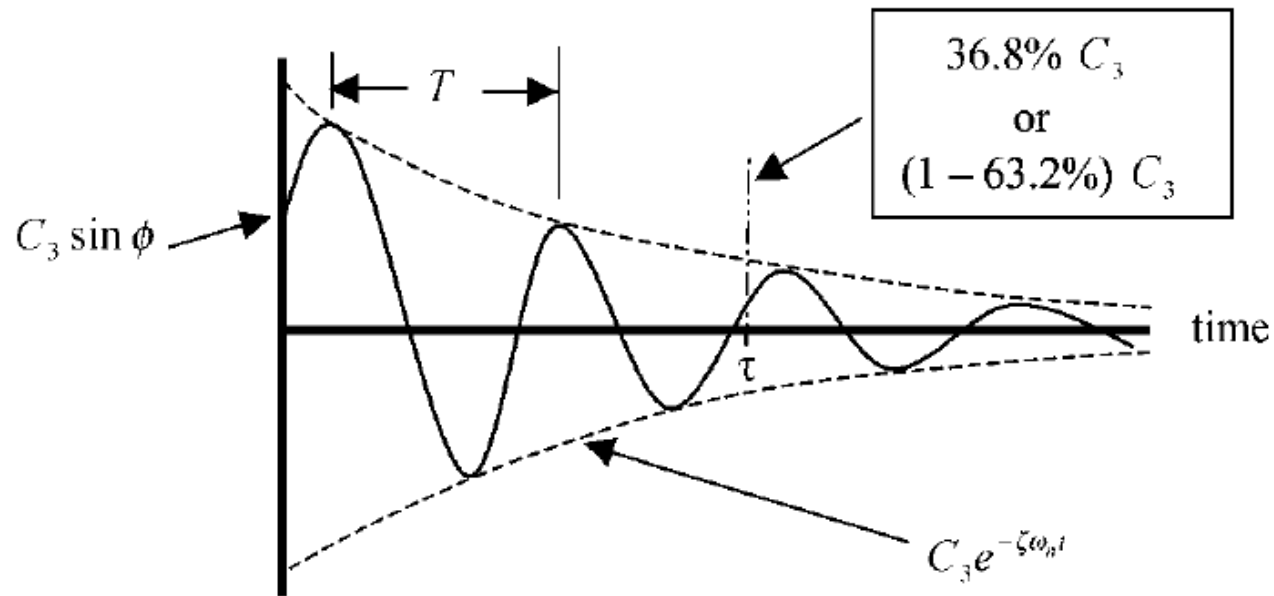
- The period of oscillation (T) for a second order system is the time it takes between consecutive peaks of an oscillation. The period is inversely proportional to the damped frequency and is defined by

$$T = \frac{2\pi}{\omega_D}$$

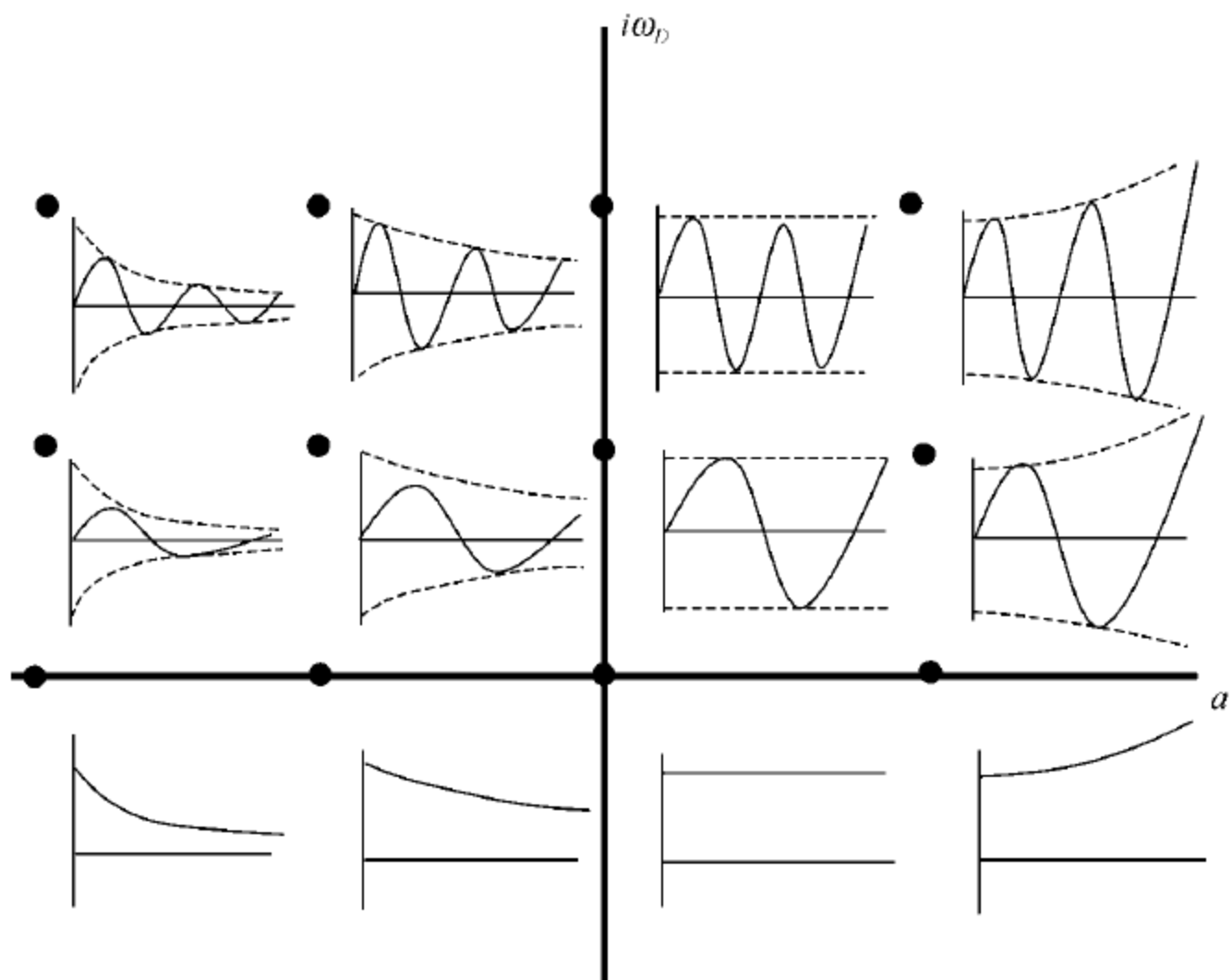
ω_D must be in rad / sec

- The time response for the homogeneous equation

$$\ddot{x} + 2\xi\omega_N\dot{x} + \omega_N^2x = \omega_N^2$$



$$x(t) = C_3 e^{-\xi\omega_N t} \sin(\omega_D t + \phi)$$



Thank you